

# Strong Convergence Rate of Finite Difference Approximations for Stochastic Cubic Schrödinger Equations

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**ABSTRACT.** We derive a strong convergence rate of spatial finite difference approximation for both focusing and defocusing stochastic cubic Schrödinger equations with a multiplicative  $Q$ -Wiener process. Beyond the uniform boundedness of moments of high order derivatives of the exact solution, the key requirement of our approach is the exponential integrability of both the exact and numerical solutions. By constructing and analyzing a Lyapunov functional, we derive the uniform boundedness of moments of high order derivatives of the exact solution. The latter exponential integrability is obtained by a version of a criterion given by [Cox, Hutzenthaler and Jentzen, arXiv:1309.5595]. As a by-product, we prove for the first time that the solution of this equation depends continuously on the initial data and obtain a large deviation-type result on the dependence of the noise with first order strong convergence rate.

## 1. Introduction and main idea

Strong convergence rates are particularly important for efficient multilevel Monte Carlo methods (see e.g. [2, 17]). There is a general theory on strong error estimates for stochastic partial differential equations (SPDEs) with Lipschitz coefficients (see e.g. [1, 3, 5, 13] and references therein), where one usually adopt the semi-group or equivalent Green's function framework. For SPDEs with non-Lipschitz but monotone coefficients, one can use the variational framework to derive a strong convergence rate of numerical approximations (see e.g. [18]). Unfortunately, the monotonicity assumption is too restrictive in the sense that the coefficients of the majority of nonlinear SPDEs from applications, including stochastic Navier-Stokes equations, stochastic Burgers equations, Cahn-Hilliard-Cook equations, stochastic nonlinear wave equations and stochastic nonlinear Schrödinger (NLS) equations etc., do not satisfy the monotonicity assumption.

Our main purpose in this paper is to derive a strong convergence rate of spatial approximations for one-dimensional stochastic cubic Schrödinger equation

$$idu + (\Delta u + \lambda |u|^2 u)dt = u \circ dW(t) \quad \text{in } (0, T] \times \mathcal{O}; \quad u(0) = u_0$$

under homogenous Dirichlet boundary condition, where  $\lambda = 1$  or  $-1$  correspond to focusing or defocusing case,  $T \in (0, \infty)$ ,  $\mathcal{O} = (0, 1)$  and  $W = \{W(t) : t \in [0, T]\}$

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is a  $Q$ -Wiener process on a stochastic basis  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ , i.e., there exists an real-valued, orthonormal basis  $\{e_k\}_{k=1}^\infty$  of  $L_2(\mathcal{O})$  and a sequence of mutually independent, real-valued Brownian motions  $\{\beta_k\}_{k=1}^\infty$  such that  $W(t) = \sum_{k=1}^\infty Q^{\frac{1}{2}} e_k \beta_k(t)$  for  $t \in [0, T]$ . For convenience, we always consider the equivalent Itô equation

$$(1.1) \quad du = (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - uF_Q/2) dt - \mathbf{i}u dW(t) \quad \text{in } (0, T] \times \mathcal{O}; \quad u(0) = u_0,$$

where  $F_Q := \sum_{k=1}^\infty (Q^{\frac{1}{2}} e_k)^2$ . Many authors concern numerical approximations for Eq. (1.1) (see e.g. [6, 7, 10, 11] and references therein). To deal with the nonlinearity, one usually apply the truncated technique. However, this idea only produce pathwise convergence rates or convergence rates in probability. In this paper, by deducing the uniform boundedness for moments of high order derivatives of the exact solution and exponential moments of both the exact and numerical solutions, we obtain the strong error estimate of the center difference scheme which is a representative finite difference scheme. To the best of our knowledge, this is the first result deriving strong convergence rates, i.e., rates in  $L^p(\Omega)$  with some  $p \in [1, \infty)$ , of numerical approximations for Eq. (1.1). For other types of SPDEs, we are only aware that [12] analyze the spectral Galerkin approximations for the two-dimensional stochastic Navier-Stokes equation using a uniform bound for exponential moments of the solution derived in [15], and [16] study the spectral Galerkin approximations for the one-dimensional Cahn-Hilliard-Cook equation and stochastic Burgers equation through exponential integrability characterized in [8].

To propose our main idea, we need to introduce some frequently used notations and related technical analytic tools.

### 1.1. Notations and analytic tools.

- (1) Denote  $H := L_2(\mathcal{O})$  with inner product  $\langle f, g \rangle_H := \Re [\int_{\mathcal{D}} \bar{f}(x)g(x)dx]$  for complex-valued function  $f, g \in H$  and  $l_{2,h}$  with inner product  $\langle f, g \rangle_h := \sum_{l=0}^N \Re [f(l)g(l)] h$ . We also use the discrete  $l_{4,h}$  and  $l_\infty$  spaces with norms  $\|f\|_{4,h} := (\sum_{l=0}^N |f(l)|^4 h)^{1/4}$  and  $\|f\|_\infty := \sup_{l \in \{0,1,\dots,N+1\}} |f(l)|$ .
- (2) Denote by  $\mathcal{L}_2(H, \tilde{H})$  the space of Hilbert-Schmidt operators  $\Psi$  from  $H$  to another separable Hilbert space  $\tilde{H}$ , endowed with the norm  $\|\Psi\|_{\mathcal{L}_2(H, \tilde{H})} := (\text{tr}(\Psi^* \Psi))^{1/2} = (\sum_{k=1}^\infty \|\Psi e_k\|_{\tilde{H}}^2)^{1/2}$ , where  $\{e_k\}_{k=1}^\infty$  is any orthonormal basis of  $H$ . In particular,  $\mathcal{L}_2(H, H^\delta)$  is denoted by  $\mathcal{L}_2^\delta$ . Throughout the paper, we assume that  $Q^{\frac{1}{2}} \in \mathcal{L}_2^\delta$  for certain  $\delta$ .
- (3) Let  $0 = x_0 < x_1 < \dots < x_{N+1} = 1$  be uniform partition of the interval  $\mathcal{O}$  with the step size  $h = \frac{1}{N+1}$ . For a grid function  $f^h$ , we denote  $f^h(x_l) = f^h(l)$  for simplicity. We use  $D_+$  and  $D_-$  to denote the forward difference operator and backward difference operator, respectively, i.e.,  $D_+ f^h(l) := \frac{f^h(l+1) - f^h(l)}{h}$  and  $D_- f^h(l) := \frac{f^h(l) - f^h(l-1)}{h}$ .
- (4) To bound the  $\|\cdot\|_{L^\infty}$ -norm, we need the Gagliardo-Nirenberg inequality

$$(1.2) \quad \|f\|_{L^\infty}^2 \leq 2\|f\| \cdot \|\nabla f\|$$

for  $f \in H_0^1$  and its discrete correspondence

$$(1.3) \quad \|f^h\|_\infty^2 \leq 2\|f^h\|_h \|D_+ f^h\|_h$$

for any grid function  $f^h$  vanishing on  $\partial\mathcal{O}$ . We also use the Sobolev embedding  $H^\delta \hookrightarrow L^\infty$  for  $\delta > 1/2$ , and its consequence which shows that  $H^\delta$

is an algebra when  $\delta > 1/2$ , i.e., for any  $f, g \in H^\delta$  with  $\delta > 1/2$ , there exists  $C = C(\delta)$  such that

$$(1.4) \quad \|fg\|_{H^\delta} \leq C\|f\|_{H^\delta}\|g\|_{H^\delta}.$$

Throughout  $C$  is a generic constant which will different from line to line.

**1.2. Main idea.** Our main aim is to derive the convergence of the spatial center difference scheme

$$(1.5) \quad du^h(l) = \left( \mathbf{i}D_+D_-u^h(l) + \mathbf{i}\lambda|u^h(l)|^2u^h(l) - u^h(l)F_Q(l)/2 \right) dt - \mathbf{i}u^h(l)dW(t, l)$$

towardr Eq. (1.1) with algebraic rate in strong sense. Define  $R_h(l) := \Delta u(l) - D_+D_-u(l)$  for  $l \in \{0, 1, \dots, N, N+1\}$ . The initial data of Eq. (1.1) and Eq. (1.5) are  $u_0$  and grid function  $u^h(0, l) = u_0(l)$ ,  $l \in \{0, 1, \dots, N, N+1\}$ , respectively. It is clear that Eq. (1.1) and Eq. (1.5) possess the continuous and discrete charge conservation laws, respectively, i.e., for all  $t \in [0, T]$  it holds a.s. that

$$(1.6) \quad \|u(t)\|_H^2 = \|u_0\|_H^2, \quad \|u^h(t)\|_h^2 = \|u_0\|_h^2.$$

The exact solution of Eq. (1.1), at the grid points, satisfies

$$du(l) = \left( \mathbf{i}D_+D_-u(l) + \mathbf{i}R_h(l) + \mathbf{i}\lambda|u(l)|^2u(l) - u(l)F_Q(l)/2 \right) dt - \mathbf{i}u(l)dW(t, l).$$

Denote  $\epsilon^h(l) = u(l) - u^h(l)$ . Applying Itô formula to the functional  $f(X(t)) := \|X(t)\|_h^2$  with  $X(t) = \epsilon^h(t)$ , using the continuous and discrete Gagliardo-Nirenberg inequalities (1.2)–(1.3) and charge conservation laws (1.6) (more details see Theorem 4.1), we obtain

$$(1.7) \quad \|\epsilon^h(t)\|_h^2 \leq \int_0^t \|R^h\|_\infty^2 dr + \int_0^t \left( 1 + 2\|u_0\|_H\|\nabla u\|_H + 2\|u_0^h\|_h\|D_+u^h\|_h \right) \|\epsilon^h\|_h^2 dr.$$

Taking expectation, as in the deterministic case, leads to

$$\begin{aligned} & \mathbb{E} [\|\epsilon^h(t)\|_h^2] \\ & \leq \int_0^t \mathbb{E} [\|R^h\|_\infty^2] dr + \int_0^t \mathbb{E} \left[ \left( 1 + 2\|u_0\|_H\|\nabla u\|_H + 2\|u_0^h\|_h\|D_+u^h\|_h \right) \|\epsilon^h\|_h^2 \right] dr. \end{aligned}$$

Due to the appearance of the nonlinear term in the last integral above, the Gronwall inequality is not available and one can't derive a strong convergence rate. These difficulties are common features to obtain a strong convergence rate for numerical approximations appearing in many situations, see e.g. [7, 11] for stochastic NLS equations and [4, 19] for other SPDEs with non-monotone coefficients.

Our main idea is applying Gronwall-Bellman inequality to (1.7) before taking expectation. Then using Hölder and Minkowski inequalities, we have

$$(1.8) \quad \left\| \mathbb{E} \left[ \sup_{t \in [0, T]} \|\epsilon^h(t)\|_h^2 \right] \right\|_{L^s(\Omega)} \leq T^{\frac{1}{2}} e^{\frac{T}{2}} \left\| \mathbb{E} \left[ \sup_{t \in [0, T]} \|R^h\|_\infty^4 \right] \right\|_{L^s(\Omega)}^{\frac{1}{4}} \left\| \exp \left( \int_0^T \|u_0\|_H\|\nabla u\|_H dr \right) \right\|_{L^s(\Omega)} \left\| \exp \left( \int_0^T \|u_0^h\|_h\|D_+u^h\|_h dr \right) \right\|_{L^s(\Omega)}.$$

In order to obtain the strong convergence rate for our scheme (1.5), we need to estimate these three terms on the right side of (1.8). The first expectation produces strong convergence rate  $\mathcal{O}(h^2)$  with multiple which can be controlled by the  $p$ -moments of the solution under  $H^s$ -norm, which is proved to be uniformly bounded in Theorem 2.1 and Corollary 2.1. Here and after,  $s$  denotes an integer. To bound

the last two exponential moments with stochastic initial datum  $u_0$ , we apply a revision of a criterion given by [8] on exponential integrability of an Hilbert-valued stochastic process which is the strong solution of an SDE in Hilbert space (see Lemma 3.1 and Proposition 3.1). Meanwhile, we derive the continuous dependence for the solution of Eq. (1.1) on both the initial data and the noise with explicit rate (see Corollaries 3.1 and 3.2).

The paper is organized as follows. In Section 2, we bound the  $p$ -moments of high order derivatives of the solution and discrete first derivative of the discrete solution. The uniform bound on exponential moments of energy functional of solution is proved in Section 3. The results in Section 2 and Section 3 are used in Section 4 to derive the strong convergence rate of the finite difference approximation.

## 2. Well-posedness and regularity

In this Section, we first prove the moment's uniform boundedness of the continuous equation (1.1), which is necessary for proving the global well-posedness of Eq. (1.1) and estimating the strong error. Then we show a priori estimate and thus the well-posedness of the discrete equation (1.5).

For  $s = 1$  or  $2$ , the solution  $u$  is in  $H^s$  under some assumptions on  $u_0$  and  $Q$  (see [9] for  $s = 1$  and see [7] for  $s = 2$ ). Our main result in this part is the stability of  $u$  in  $H^s$  with integer  $s \geq 2$  for both focusing and defocusing cases. This will enables us to bound  $\mathbb{E} [\|R^h\|_\infty^4]$  appearing in (1.8). We remark that our proof can be also applied to the whole line, including the focusing case beyond the defocusing case established in [7].

Throughout this section, we assume that  $u_0 \in H_0^{s-1} \cap H^s$  for  $s = 2$  a.s. We construct a Lyapunov functional

$$(2.1) \quad f(u) = \|\nabla^s u\|_H^2 - \lambda \langle (-\Delta)^{s-1} u, |u|^2 u \rangle_H$$

and study its evolution. Simple calculations yield that the first order derivative of  $f(u)$  is

$$(2.2) \quad \begin{aligned} Df(u)(v) &= 2\langle \nabla^s u, \nabla^s v \rangle_H - 2\lambda \langle (-\Delta)^{s-1} u, u \Re[\bar{u}v] \rangle_H \\ &\quad - \lambda \langle (-\Delta)^{s-1} u, |u|^2 v \rangle_H - \lambda \langle (-\Delta)^{s-1} v, |u|^2 u \rangle_H, \quad \forall v \in \mathcal{C}_0^\infty, \end{aligned}$$

and its second order derivative is

$$(2.3) \quad \begin{aligned} D^2 f(u)(v, w) &= 2\langle \nabla^s v, \nabla^s w \rangle_H - 2\lambda \langle (-\Delta)^{s-1} u, w \Re[\bar{u}v] \rangle_H \\ &\quad - 2\lambda \langle (-\Delta)^{s-1} w, u \Re[\bar{u}v] \rangle_H - 2\lambda \langle (-\Delta)^{s-1} u, u \Re[\bar{v}w] \rangle_H \\ &\quad - 2\lambda \langle (-\Delta)^{s-1} u, v \Re[\bar{u}w] \rangle_H - \lambda \langle (-\Delta)^{s-1} w, |u|^2 v \rangle_H \\ &\quad - 2\lambda \langle (-\Delta)^{s-1} v, u \Re[\bar{u}w] \rangle_H - \lambda \langle (-\Delta)^{s-1} v, |u|^2 w \rangle_H, \quad \forall v, w \in \mathcal{C}_0^\infty. \end{aligned}$$

Some calculations in the following are formal. To make them rigorous, we need some truncated skill and Garlerkin approximations as in [9].

**THEOREM 2.1.** Assume that  $u_0 \in \bigcap_{m=2}^s L^{3^{s-m}p}(\Omega; H^m) \cap \bigcap_{m=0}^1 L^{3^{s-m-1}5p}(\Omega; H^m)$

and  $Q^{\frac{1}{2}} \in \mathcal{L}_2^s$  for some  $p \in [2, \infty)$  and  $s \geq 2$ . There exists  $C = C(p, T, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}) \in (0, \infty)$  such that

$$(2.4) \quad \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^s}^p] \leq C.$$

PROOF. We first present the proof of (2.4) for  $p = 2$ . Applying Itô formula to the functional  $f(u)$  defined by (2.1), we get

$$\begin{aligned}
 f(u(t)) - f(u_0) &= \int_0^t Df(u) (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - uF_Q/2) dr \\
 &\quad + \frac{1}{2} \int_0^t \text{tr} \left[ D^2 f(u) (-\mathbf{i}uQ^{\frac{1}{2}})(-\mathbf{i}uQ^{\frac{1}{2}})^* \right] dr \\
 (2.5) \quad &\quad + \int_0^t Df(u)(-\mathbf{i}u)dW(r) =: I_1(t) + I_2(t) + I_3(t).
 \end{aligned}$$

By (1.4) and integration by parts, we have

$$(2.6) \quad |\mathbb{E}[f(u_0)]| \leq C \left( \mathbb{E} [|u_0|_{H^s}^2] + \mathbb{E} [\|u_0\|_{H^{s-1}}^4] \right).$$

We first estimate  $I_1(t)$ . Substituting the expression (2.2) of  $Df$  into  $I_1(t)$ , we have

$$\begin{aligned}
 I_1(t) &= 2 \int_0^t \langle \nabla^s u, \nabla^s (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - uF_Q/2) \rangle_H dr \\
 &\quad - \lambda \int_0^t \langle (-\Delta)^{s-1} u, u[\mathbf{i}\bar{u}\Delta u - \mathbf{i}u\Delta\bar{u} - |u|^2F_Q] \rangle_H dr \\
 &\quad - \lambda \int_0^t \langle (-\Delta)^{s-1} u, |u|^2(\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - uF_Q/2) \rangle_H dr \\
 &\quad - \lambda \int_0^t \langle (-\Delta)^{s-1} (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2u - uF_Q/2), |u|^2u \rangle_H dr \\
 &:= I_{11}(t) + I_{12}(t) + I_{13}(t) + I_{14}(t).
 \end{aligned}$$

By integration by parts,  $I_{11}(t)$  and  $I_{12}(t)$  can be rewritten as

$$\begin{aligned}
 I_{11}(t) &= -2 \int_0^t \langle (-\Delta)^{s-1} u, \mathbf{i}\lambda\Delta(|u|^2u) \rangle_H dr - \int_0^t \langle \nabla^s u, \nabla^s (uF_Q) \rangle_H dr \\
 &=: I_{111}(t) + I_{112}(t), \\
 I_{12}(t) &= -\lambda \int_0^t \langle (-\Delta)^{s-1} u, u(\mathbf{i}\bar{u}\Delta u - \mathbf{i}u\Delta\bar{u}) \rangle_H dr + \lambda \int_0^t \langle (-\Delta)^{s-1} u, |u|^2uF_Q \rangle_H dr \\
 &=: I_{121}(t) + I_{122}(t).
 \end{aligned}$$

Cauchy-Schwarz inequality and the estimate (1.4) yield that

$$|\mathbb{E}[I_{112}(t)]| \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \left( \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^{s-1}}^2] T + \int_0^t \mathbb{E} [|u|_{H^s}^2] dr \right).$$

Similarly,

$$|\mathbb{E}[I_{122}(t)]| \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{s-1}}^2 \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^{s-1}}^4] T.$$

Divide  $I_{111}(t)$  into two equal parts which can balance  $I_{121}(t)$  and  $I_{14}(t)$ . More precisely, inserting the identities  $\Delta(|u|^2u) = 2|u|^2\Delta u + 4u|\nabla u|^2 + 2\bar{u}(\nabla u)^2 + u^2\Delta\bar{u}$

and  $\langle \Delta^{s-1}(|u|^2 u), \mathbf{i} \Delta u \rangle_H + \langle \Delta^{s-1} u, \mathbf{i} \Delta(|u|^2 u) \rangle_H = 0$ , we have

$$\begin{aligned} & I_{111}(t) + I_{121}(t) + I_{14}(t) \\ &= (I_{111}(t)/2 + I_{121}(t)) + (I_{111}(t)/2 + I_{14}(t)) \\ &= -\lambda \int_0^t \langle (-\Delta)^{s-1} u, 3\mathbf{i}|u|^2 \Delta u \rangle_H dr - \lambda \int_0^t \langle (-\Delta)^{s-1} u, 4\mathbf{i}|\nabla u|^2 u + 2\mathbf{i}(\nabla u)^2 \bar{u} \rangle_H dr \\ &\quad - \lambda \int_0^t \langle (-\Delta)^{s-1}(|u|^2 u), -u F_Q/2 \rangle_H dr =: I_a(t) + I_b(t) + I_c(t). \end{aligned}$$

Applying integration by parts, Leibniz formula and the fact that  $\langle \nabla^s u, \mathbf{i}|u|^2 \nabla^s u \rangle_H = 0$ , we have

$$\begin{aligned} \langle (-\Delta)^{s-1} u, \mathbf{i}|u|^2 \Delta u \rangle_H &= \langle \nabla^s u, \mathbf{i} \nabla^{s-2}(|u|^2 \cdot \Delta u) \rangle_H \\ &= \sum_{j=0}^{s-3} C_{s-2}^j \langle \nabla^s u, \nabla^{s-2-j} |u|^2 \cdot \nabla^j(\Delta u) \rangle_H. \end{aligned}$$

It follows from the above estimate, Sobolev embedding  $H^1 \hookrightarrow L^\infty$  and (1.4) that

$$|\langle (-\Delta)^{s-1} u, \mathbf{i}|u|^2 \Delta u \rangle_H| \leq C|u|_{H^s} \sum_{j=0}^{s-3} \|\nabla^{s-2-j} |u|^2\|_{L^\infty} \|u\|_{H^{j+2}} \leq C|u|_{H^s} \|u\|_{H^{s-1}}^3$$

for  $s > 2$ . It is also valid for  $s = 2$ , since  $\langle \Delta u, \mathbf{i}|u|^2 \Delta u \rangle_H = 0$ . This implies that

$$|\mathbb{E}[I_a(t)]| \leq C \left( \sup_{t \in [0, T]} \mathbb{E}[\|u\|_{H^{s-1}}^6] T + \int_0^t \mathbb{E}[|u|_{H^s}^2] dr \right).$$

Applying Hölder inequality, integration by parts and (1.4), we obtain

$$|\mathbb{E}[I_b(t)]| \leq C \left( \sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^{s-1}}^6] T + \int_0^t \mathbb{E}[|u|_{H^s}^2] dr \right),$$

for  $s > 2$ . When  $s = 2$ , by using the Sobolev embedding  $H^1 \hookrightarrow L^\infty$ , (1.2) with  $f = \nabla u$  and Young inequality, we get

$$\begin{aligned} |\mathbb{E}[I_b(t)]| &\leq \int_0^t \mathbb{E}[|u|_{H^2} \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} \|\nabla u\|] dr \\ &\leq C \left( \sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^1}^{10}] T + \int_0^t \mathbb{E}[|u|_{H^2}^2] dr \right). \end{aligned}$$

Similar arguments imply that

$$|\mathbb{E}[I_c(t)]| \leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{s-1}}^2 \sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^{s-1}}^4] T.$$

As a result,

$$\begin{aligned} & |\mathbb{E}[I_{111}(t) + I_{121}(t) + I_{14}(t)]| \\ &\leq C \left( \left(1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2\right) \left(1 + \sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^{s-1}}^{10}]\right) T + \int_0^T \mathbb{E}[|u|_{H^s}^2] dr \right). \end{aligned}$$

For  $I_{13}(t)$ , using integration by parts and Sobolev embedding (1.4), we have

$$\begin{aligned} |\mathbb{E}[I_{13}(t)]| &\leq C \left( \mathbb{E}[|I_a(t)|] + \int_0^t \mathbb{E} [|\langle (-\Delta)^{s-1} u, \mathbf{i}|u|^4 u + |u|^2 u F_Q/2 \rangle_H|] dr \right) \\ &\leq C \left( \left(1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2\right) \left(1 + \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^{s-1}}^6]\right) T + \int_0^T \mathbb{E} [|u|_{H^s}^2] dr \right). \end{aligned}$$

Combining the estimations of  $I_{1,1}$  to  $I_{1,4}$ , we have for any  $t \in [0, T]$ ,

$$(2.7) \quad |\mathbb{E}[I_1(t)]| \leq C \left( \left(1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2\right) \left(1 + \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^{s-1}}^6]\right) T + \int_0^T \mathbb{E} [|u|_{H^s}^2] dr \right)$$

for  $s > 2$  and

$$(2.8) \quad |\mathbb{E}[I_1(t)]| \leq C \left( \left(1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2\right) \left(1 + \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^1}^{10}]\right) T + \int_0^T \mathbb{E} [|u|_{H^2}^2] dr \right).$$

Now we turn to the estimation of  $I_2$ . Substituting  $D^2 f(u)$  given by (2.3) and noting the fact that  $\Re[\bar{u}v] = \Re[\bar{u}\omega] = 0$  for  $v = \omega = -\mathbf{i}uQ^{\frac{1}{2}}$ , we have that

$$\begin{aligned} I_2(t) &= \int_0^t \text{tr} \left[ \nabla^s (-\mathbf{i}uQ^{\frac{1}{2}})^* \otimes \nabla^s (-\mathbf{i}uQ^{\frac{1}{2}}) \right] dr \\ &\quad - 2\lambda \int_0^t \text{tr} \left[ ((-\Delta)^{s-1} \bar{u}) u \Re \left[ (-\mathbf{i}uQ^{\frac{1}{2}})^* \otimes (-\mathbf{i}uQ^{\frac{1}{2}}) \right] \right] dr \\ &\quad - \lambda \int_0^t \text{tr} \left[ (-\Delta)^{s-1} (-\mathbf{i}uQ^{\frac{1}{2}})^* \otimes |u|^2 (-\mathbf{i}uQ^{\frac{1}{2}}) \right] dr. \end{aligned}$$

Using Cauchy-Schwarz inequality and (1.4), we have for any  $t \in [0, T]$ ,

$$(2.9) \quad |\mathbb{E}[I_2(t)]| \leq C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \left( \left(1 + \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^{s-1}}^6]\right) T + \int_0^T \mathbb{E} [|u|_{H^s}^2] dr \right).$$

On the other hand, owing to the property of Itô integral,

$$(2.10) \quad \mathbb{E}[I_3(t)] = 0.$$

For  $p \in [2, \infty)$ , we apply Itô formula to  $f^{\frac{p}{2}}(u)$  and obtain

$$\begin{aligned} f^{\frac{p}{2}}(u(t)) &= f^{\frac{p}{2}}(u_0) + \int_0^t \frac{p}{2} f^{\frac{p}{2}-1}(u) Df(u) (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - u F_Q/2) dr \\ &\quad + \frac{1}{2} \int_0^t \frac{p}{2} \left( \frac{p}{2} - 1 \right) \text{tr} \left[ f^{\frac{p}{2}-2}(u) D^2 f(u) (-\mathbf{i}uQ^{\frac{1}{2}}) (-\mathbf{i}uQ^{\frac{1}{2}})^* \right] dr \\ (2.11) \quad &+ \int_0^t \frac{p}{2} f^{\frac{p}{2}-1}(u) Df(u) (-\mathbf{i}u) dW(r). \end{aligned}$$

It follows from (1.4) and Cauchy-Schwarz inequality that  $f^{\frac{p}{2}-1}(u) \leq C(\|u\|_{H^{s-1}}^{2(p-2)} + |u|_{H^s}^{p-2})$ . By the estimation of  $I_1(t)$ , it holds that a.s.

$$|Df(u)(\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - u F_Q/2)| \leq C \left( (1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2) \|u\|_{H^{s-1}}^6 + |u|_{H^s}^2 \right)$$

for  $s > 2$  and

$$|Df(u)(\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - u F_Q/2)| \leq C \left( (1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^2}^2) \|u\|_{H^1}^{10} + |u|_{H^2}^2 \right).$$

By Young inequality, we get an estimate for the first integral in (2.11):

$$\begin{aligned}
& \mathbb{E} \left[ \left| \int_0^t \frac{p}{2} f^{\frac{p}{2}-1}(u) Df(u) (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - uF_Q/2) dr \right| \right] \\
& \leq C \left( \int_0^t \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \mathbb{E} \left[ \|u\|_{H^{s-1}}^{2p+2} \right] + \mathbb{E} \left[ \|u\|_{H^{s-1}}^{2p-4} |u|_{H^s}^2 \right] \right. \\
& \quad \left. + \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \mathbb{E} \left[ \|u\|_{H^{s-1}}^6 |u|_{H^s}^{p-2} \right] + \mathbb{E} [|u|_{H^s}^p] dr \right) \\
& \leq C \left( \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^{s-1}}^{3p} \right] \right) T + \int_0^t \mathbb{E} [|u|_{H^s}^p] dr \right)
\end{aligned}$$

for  $s > 2$  and

$$\begin{aligned}
& \mathbb{E} \left[ \left| \int_0^t \frac{p}{2} f^{\frac{p}{2}-1}(u) Df(u) (\mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - uF_Q/2) dr \right| \right] \\
& \leq C \left( \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^1}^{5p} \right] \right) T + \int_0^t \mathbb{E} [|u|_{H^2}^p] dr \right).
\end{aligned}$$

Similar arguments can be applied to other terms of  $f^{\frac{p}{2}}(u(t))$  in (2.11). Thus we obtain

$$\begin{aligned}
\mathbb{E} [|u(t)|_{H^s}^p] & \leq C \mathbb{E} \left[ f^{\frac{p}{2}}(u(t)) \right] + C \mathbb{E} \left[ \|u\|_{H^{s-1}}^{2p} \right] \\
& \leq C \left( 1 + \mathbb{E} [\|u_0\|_{H^s}^p] + \mathbb{E} [\|u_0\|_{H^{s-1}}^{2p}] \right) + C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \\
& \quad \times \left( \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^{s-1}}^{3p} \right] \right) T + \int_0^T \mathbb{E} [\|u\|_{H^s}^p] dr \right)
\end{aligned}$$

for  $s > 2$  and

$$\begin{aligned}
\mathbb{E} [|u(t)|_{H^2}^p] & \leq C \left( 1 + \mathbb{E} [\|u_0\|_{H^2}^p] + \mathbb{E} [\|u_0\|_{H^1}^{2p}] \right) + C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \\
& \quad \times \left( \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^1}^{5p} \right] \right) T + \int_0^T \mathbb{E} [\|u\|_{H^s}^p] dr \right).
\end{aligned}$$

Gronwall inequality then leads to

$$\begin{aligned}
\mathbb{E} [|u(t)|_{H^s}^p] & \leq C \exp \left( C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) T \right) \left[ \left( 1 + \mathbb{E} [\|u_0\|_{H^s}^p] + \mathbb{E} [\|u_0\|_{H^{s-1}}^{2p}] \right) \right. \\
& \quad \left. + \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^{s-1}}^{3p} \right] \right) T \right]
\end{aligned}$$

for  $s > 2$  and

$$\begin{aligned}
\mathbb{E} [|u(t)|_{H^2}^p] & \leq C \exp \left( C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) T \right) \left[ \left( 1 + \mathbb{E} [\|u_0\|_{H^2}^p] + \mathbb{E} [\|u_0\|_{H^1}^{2p}] \right) \right. \\
& \quad \left. + \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^1}^{5p} \right] \right) T \right].
\end{aligned}$$

Similar arguments as Theorems 4.1 and 4.6 in [9] lead to  $\sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^1}^{5p}] < \infty$  provided that  $u_0 \in L^{5p}(\Omega; H^1) \cap L^{15p}(\Omega; H)$ . This implies that  $\sup_{t \in [0, T]} \mathbb{E} [|u(t)|_{H^2}^p] < \infty$



$\infty$  provided that  $u_0 \in L^p(\Omega; H^2) \cap L^{5p}(\Omega; H^1) \cap L^{15p}(\Omega; H)$ . For  $s = 3$ , when  $u_0 \in L^p(\Omega; H^3) \cap L^{3p}(\Omega; H^2) \cap L^{15p}(\Omega; H^1) \cap L^{45p}(\Omega; H)$ , it holds that

$$\begin{aligned} \mathbb{E} [|u(t)|_{H^3}^p] &\leq C \exp \left( C \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^3}^2 \right) T \right) \left[ \left( 1 + \mathbb{E} [\|u_0\|_{H^3}^p] + \mathbb{E} [\|u_0\|_{H^2}^{2p}] \right) \right. \\ &\quad \left. + \left( 1 + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^3}^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} [\|u(t)\|_{H^2}^{3p}] \right) T \right] < \infty. \end{aligned}$$

By induction, we complete the proof of (2.4).  $\square$

REMARK 2.1. In [9], Theorem 4.6, a uniform bound for the Hamiltonian is used to construct a unique solution with continuous  $H^1(\mathbb{R}^d)$ -valued paths for Eq. (1.1). We can follow the same strategy in [9] to construct the unique local mild solution with continuous  $H^s(\mathcal{O})$ -valued paths by a contraction argument, and then show that it is global by the a priori estimate of  $f(u)$  with  $s \geq 2$ . To prove this mild solution is also a strong one of Eq. (1.1), we refer to [20], Propositions F.0.4 and F.0.5.

COROLLARY 2.1. Under the same conditions of Theorem 2.1 for  $p \in [4, \infty)$  and  $s \geq 2$ , there exists  $C = C(T, p, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}) \in (0, \infty)$  such that

$$(2.12) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t)\|_{H^s}^p \right] \leq C.$$

PROOF. The main step is that we need to deal with the stochastic integral  $I_3(t)$  which is vanished in Theorem 2.1. By the expression of  $Df(u)$ , we get

$$\begin{aligned} I_3(t) &= 2 \int_0^t \langle \nabla^s u, \nabla^s (-i u dW(r)) \rangle_H - \lambda \int_0^t \langle (-\Delta)^{s-1} u, |u|^2 (-i u dW(r)) \rangle_H \\ &\quad - \lambda \int_0^t \langle (-\Delta)^{s-1} (-i u dW(r)), |u|^2 u \rangle_H. \end{aligned}$$

For  $p \in [4, \infty)$ , applying Burkholder-Davis-Gundy and Hölder inequalities, we obtain

$$\begin{aligned} &\mathbb{E} \left[ \sup_{t \in [0, T]} |I_3(t)|^{\frac{p}{2}} \right] \\ &\leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^{\frac{p}{2}} \left( \mathbb{E} \left[ \left| \int_0^T |u|_{H^s}^2 \|u\|_{H^{s-1}}^2 dr \right|^{\frac{p}{4}} \right] + \mathbb{E} \left[ \left| \int_0^T |u|_{H^s}^2 \|u\|_{H^{s-2}}^6 dr \right|^{\frac{p}{4}} \right] \right) \\ &\leq C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^s}^{\frac{p}{2}} \left( \left( 1 + \mathbb{E} \left[ \sup_{t \in [0, T]} |u|_{H^{s-1}}^{3p} \right] \right) T + \int_0^T \mathbb{E} [|u|_{H^s}^p] dr \right). \end{aligned}$$

Applying the same induction arguments as in Theorem 2.1, we get (2.12).  $\square$

The local existence and uniqueness of the solution for Eq. (1.5) can be proved by the contraction argument in [9] for Eq. (1.1). The global existence is an immediate consequence of the following a priori estimate. To this end, we define the discrete energy functional

$$(2.13) \quad U^h(u^h) := \frac{1}{2} \|D_+ u^h\|_h^2 - \frac{\lambda}{4} \|u^h\|_{4,h}^4.$$

PROPOSITION 2.1. Assume that  $u_0^h \in L^{3p}(\Omega, l_{2,h})$ ,  $D_+ u_0^h \in L^p(\Omega, l_{2,h})$  and  $Q \in \mathcal{L}_2^{1+\delta}$  for some  $p \in [2, \infty)$  and  $\delta > 1/2$ . Then there exists  $C(p, T, u_0^h, \|Q\|_{\mathcal{L}_2^{1+\delta}}) \in (0, \infty)$  such that

$$(2.14) \quad \sup_{t \in [0, T]} \mathbb{E} [\|D_+ u^h(t)\|_h^p] \leq C.$$

Furthermore, assume that  $p \in [4, \infty)$ , then

$$(2.15) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \|D_+ u^h(t)\|_h^p \right] \leq C.$$

PROOF. We only prove (2.14) for  $p = 2$ , and similarly to the proof of Theorem 2.1 it holds for  $p \in [2, \infty)$ ; the proof for (2.15) for  $p \in [4, \infty)$  is similar to the proof of Corollary 2.1. Applying Itô formula to the energy functional  $U^h(u^h)$  defined by (2.13), we obtain

$$\begin{aligned} & U^h(u^h(t)) - U^h(u_0) \\ &= \lambda \int_0^t \langle D_+ u^h, \mathbf{i} D_+(|u^h|^2 u^h) \rangle_h dr - \sum_{k=1}^{\infty} \int_0^t \langle D_+ u^h, \mathbf{i} u^h D_+(Q^{\frac{1}{2}} e_k) \rangle_h d\beta_k(r) \\ &+ \frac{1}{2} \langle D_+ D_- u^h, u^h F_Q \rangle_h + \lambda \int_0^t \langle D_+ D_- u^h, \mathbf{i} |u^h|^2 u^h \rangle_h dr \\ &+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \|D_+(u^h Q^{\frac{1}{2}} e_k)\|_h^2 dr := U_a + U_b + U_c + U_d + U_e. \end{aligned}$$

Due to the symmetry of the numerical scheme (1.5) and the Dirichlet boundary condition, the term  $U_a + U_d$  vanishes. Then the martingale property of the Itô integral with respect to  $W$  yields that

$$\mathbb{E} [U^h(u^h(t))] = \mathbb{E} [U^h(u_0)] + \mathbb{E} [U_c + U_e].$$

By integration by parts,

$$\begin{aligned} U_c + U_e &= -\frac{1}{2} \int_0^t \left( \sum_{l=0}^N h \Re \left[ \overline{D_+ u^h(l)} D_+ u^h(l) F_Q(l+1) \right] + \langle D_+ u^h, u^h D_+ F_Q \rangle_h \right) dr \\ &+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \left( \sum_{l=0}^N h |D_+ u^h(l) Q^{\frac{1}{2}} e_k(l+1)|^2 + \|u^h D_+(Q^{\frac{1}{2}} e_k)\|_h^2 \right) dr \\ &+ \int_0^t \sum_{k=1}^{\infty} \sum_{l=0}^N h \Re \left[ \overline{D_+ u^h(l)} u^h(l) D_+(Q^{\frac{1}{2}} e_k)(l) Q^{\frac{1}{2}} e_k(l+1) \right] dr. \end{aligned}$$

Similar calculations as Theorem 2.1 deduce that

$$\begin{aligned} \mathbb{E} [U^h(u^h(t))] &\leq \mathbb{E} [U^h(u_0)] + \frac{3t}{2} \sum_{k=1}^{\infty} \|\nabla(Q^{\frac{1}{2}} e_k)\|_{L^\infty}^2 \mathbb{E} [\|u_0^h\|_h^2] \\ &\leq \mathbb{E} [U^h(u_0)] + \frac{3C_\delta^2 t}{2} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \mathbb{E} [\|u_0^h\|_h^2]. \end{aligned}$$

Cauchy-Schwarz inequality and (1.3) implies that

$$(2.16) \quad \frac{1}{4} \|D_+ u^h\|_h^2 - \frac{1}{4} \|u^h\|_h^6 \leq U^h(u^h) \leq \frac{3}{4} \|D_+ u^h\|_h^2 + \frac{1}{4} \|u^h\|_h^6.$$

As a result, we get

$$\sup_{t \in [0, T]} \mathbb{E} [U^h(u^h(t))] \leq \frac{3}{4} \mathbb{E} [\|D_+ u_0^h\|_h^2] + \frac{1}{4} \mathbb{E} [\|u_0^h\|_h^6] + C \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \mathbb{E} [\|u_0^h\|_h^2] T$$

By (1.6) and (2.16),

$$\sup_{t \in [0, T]} \mathbb{E} [\|D_+ u^h\|_h^2] \leq C \left( \mathbb{E} [\|D_+ u_0^h\|_h^2] + \mathbb{E} [\|u_0^h\|_h^6] + \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \mathbb{E} [\|u_0^h\|_h^2] T \right),$$

from which and Gronwall inequality, we complete the proof.  $\square$

### 3. Exponential integrability and continuous dependence

In this section, we establish the exponential integrability for the stochastic cubic Schrödinger equation (1.1) and its finite difference approximation (1.5). As a by-product, the exact solution depends continuously on the initial data as well as on the noise with explicit rate in  $L^p(\Omega; H)$ . For the almost surely continuous dependence in the additive case, we refer to [9].

To handle the exponential integrability for Eq. (1.1) and Eq. (1.5), we give a criterion on exponential integrability, which is a version of Corollary 2.4 in [8].

LEMMA 3.1. Let  $U \in C^2(H; \mathbb{R})$ ,  $\bar{U} \in \mathcal{L}^0(H \times \Omega; \mathbb{R})$  and  $X$  be an  $H$ -valued adapted stochastic process with continuous sample paths satisfying  $\int_0^T \|\mu(X_s)\| + \|\sigma(X_s)\|^2 dr < \infty$  a.s., and for all  $t \in [0, T]$ ,  $X_t = X_0 + \int_0^t \mu(X_s) dr + \int_0^t \sigma(X_s) dW_s$  a.s. Assume that there exists  $\mathcal{F}_0$ -measurable random variable  $\alpha \in [0, \infty)$  such that a.s.

$$(3.1) \quad DU(X)\mu(X) + \frac{\text{tr} [D^2 U(X)\sigma(X)\sigma^*(X)]}{2} + \frac{\|\sigma^*(X)DU(X)\|^2}{2e^{\alpha t}} + \bar{U}(X) \leq \alpha U(X),$$

then

$$(3.2) \quad \sup_{t \in [0, T]} \mathbb{E} \left[ \exp \left( \frac{U(X_t)}{e^{\alpha t}} + \int_0^t \frac{\bar{U}(X_r)}{e^{\alpha s}} dr \right) \right] \leq \mathbb{E} [e^{U(X_0)}].$$

PROOF. Let  $Y_t = \int_0^t \frac{\bar{U}(X_r)}{e^{\alpha r}} dr$ . Applying Itô formula to  $V(t, X_t, Y_t) := \exp \left( \frac{U(X_t)}{e^{\alpha t}} + Y_t \right)$ , we obtain

$$\begin{aligned} & V(t, X_t, Y_t) - e^{U(X_0)} \\ &= \int_0^t e^{-\alpha r} V(r, X_r, Y_r) (\bar{U}(X_r) - \alpha U(X_r)) dr + \int_0^t e^{-\alpha r} V(r, X_r, Y_r) DU(X_r) dX_r \\ & \quad + \frac{1}{2} \int_0^t e^{-\alpha r} V(r, X_r, Y_r) \text{tr} [D^2 U(X_r)\sigma(X_r)\sigma^*(X_r)] dr \\ & \quad + \frac{1}{2} \int_0^t e^{-2\alpha r} V(r, X_r, Y_r) \|\sigma^*(X_r)DU(X_r)\|^2 dr. \end{aligned}$$

Condition (3.1) implies that

$$\begin{aligned} & \mathbb{E} [V(t, X_t, Y_t)] \\ &= \mathbb{E} [\exp(U(X_0))] + \mathbb{E} \left[ \int_0^t V(r, X_r, Y_r) \left( \frac{\bar{U}(X_r) - \alpha U(X_r)}{e^{\alpha r}} + \frac{DU(X_r)\mu(X_r)}{e^{\alpha r}} \right. \right. \\ & \quad \left. \left. + \frac{\text{tr} [D^2 U(X_r)\sigma(X_r)\sigma^*(X_r)]}{2e^{2\alpha r}} + \frac{\|\sigma^*(X_r)DU(X_r)\|^2}{2e^{\alpha r}} \right) dr \right] \leq \mathbb{E} [e^{U(X_0)}]. \end{aligned}$$

This completes the proof of (3.2).  $\square$

Similar arguments of Lemma 5 and Lemma 6 in [7] yield the existence of continuous versions of the exact solution  $u$  of Eq. (1.1) and the numerical solution  $u^h$  of Eq. (1.5). To establish the exponential integrability for Eq. (1.1) and Eq. (1.5), we define

$$(3.3) \quad U(X) := \frac{1}{2} \|\nabla X\|^2 - \frac{\lambda}{4} \|X\|_{L_4}^4,$$

which is the continuous version of  $U^h$  given by (2.13).

In the rest of this section, we assume that Eq. (1.1) possesses a unique strong solution. We have the following uniform bounds of exponential-type moments for solutions of Eq. (1.1) and Eq. (1.5) from Lemma (3.1) applied to  $U$  and  $U^h$  defined by (3.3) and (2.13), respectively. This will be used in Section 4 to deduce strong convergence rate of the finite difference approximation (1.5) to Eq. (1.1).

**PROPOSITION 3.1.** Let  $p \in [1, \infty)$ . Assume that  $Q^{\frac{1}{2}} \in \mathcal{L}_2^2$  and there exist  $K_1 \in (4, \infty)$  and  $K_2 \in (3/4, \infty)$  such that

$$(3.4) \quad \mathbb{E} [K_1 \|u_0\|_H^6] + \mathbb{E} [K_2 \|\nabla u_0\|_H^2] + \mathbb{E} \left[ \exp \left( 4p^4 T^4 \exp(4C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 T) \right) \right] < \infty$$

and

$$(3.5) \quad \mathbb{E} [K_1 \|u_0^h\|_h^6] + \mathbb{E} [K_2 \|D_+ u_0^h\|_h^2] + \mathbb{E} \left[ \exp \left( 4p^4 T^4 \exp(4C_\delta^2 (\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0^h\|_h^2 T) \right) \right] < \infty.$$

There exist  $C = C(T, p, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}})$  such that

$$(3.6) \quad \left\| \exp \left( \int_0^T \|u_0\|_H \|\nabla u\|_H dr \right) \right\|_{L^p(\Omega)} \leq C,$$

$$(3.7) \quad \left\| \exp \left( \int_0^T \|u_0^h\|_h \|D_+ u^h\|_h dr \right) \right\|_{L^p(\Omega)} \leq C.$$

**PROOF.** Due to (2.16), (3.4), (3.5) and Hölder inequality, we have  $\mathbb{E} [e^{U(u_0)}] \leq C$  and  $\mathbb{E} [e^{U^h(u_0^h)}] \leq C$ . Simple calculations show that

$$\begin{aligned} DU(X)Y &= \langle \nabla X, \nabla Y \rangle_H - \lambda \langle |X|^2 X, Y \rangle_H, \\ (D^2 U(X))(Y, Z) &= \langle \nabla Z, \nabla Y \rangle_H - \lambda \langle |X|^2 Y, Z \rangle_H - 2\lambda \langle \Re[\bar{X}Y]X, Z \rangle_H. \end{aligned}$$

In the case of Eq. (1.1),  $\mu(u) = \mathbf{i}\Delta u + \mathbf{i}\lambda|u|^2 u - \frac{1}{2}uF_Q$  and  $\sigma(u) = -\mathbf{i}uQ^{\frac{1}{2}}$ . Then

$$DU(u)\mu(u) = -\frac{1}{2} \langle \nabla u, \nabla u F_Q \rangle_H - \frac{1}{2} \langle \nabla u, u \nabla F_Q \rangle_H + \frac{\lambda}{2} \langle |u|^4, F_Q \rangle_H,$$

$$\text{tr} [\sigma(u)\sigma^*(u)(D^2 U)(u)] = \sum_{k=1}^{\infty} \|\nabla(uQ^{\frac{1}{2}}e_k)\|_H^2 - \lambda \langle |u|^4, F_Q \rangle_H,$$

and

$$\|\sigma^*(u)DU(u)\|_H^2 = \langle \nabla u, -\mathbf{i}u \sum_{k=1}^{\infty} \nabla(Q^{\frac{1}{2}}e_k) \rangle_H^2.$$

Therefore, by the Sobolev inequality  $\|u\|_{L^\infty} \leq C_\delta \|u\|_{H^\delta}$ , we have

$$\begin{aligned} & DU(X)\mu(X) + \frac{1}{2} \text{tr} [\sigma(X)\sigma^*(X)D^2U] + \frac{1}{2e^{\alpha t}} \|\sigma^*(u)DU(u)\|^2 \\ &= \frac{1}{2} \left\langle |u|^2, \sum_{k=1}^{\infty} (\nabla Q^{\frac{1}{2}} e_k)^2 \right\rangle_H + \frac{1}{2e^{\alpha t}} \langle \nabla u, -iu \sum_{k=1}^{\infty} \nabla (Q^{\frac{1}{2}} e_k) \rangle_H^2 \\ &\leq \frac{C_\delta^2}{2} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 + \frac{C_\delta^2}{2} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 \|\nabla u\|_H^2. \end{aligned}$$

We conclude that for  $\lambda = -1$ ,

$$DU(u)\mu(u) + \frac{\text{tr} [\sigma(u)\sigma^*(u)D^2U(u)]}{2} + \frac{\|\sigma^*(u)DU(u)\|^2}{2e^{\alpha t}} \leq \alpha_{-1}U(u) + \beta_{-1}$$

with  $\alpha_{-1} = C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2$  and  $\beta_{-1} = \frac{C_\delta^2}{2} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2$ . When  $\lambda = 1$ , using the fact that  $\|u\|_{L^4}^4 \leq 2\|u\|_H^3 \|\nabla u\|_H$ , we deduce

$$DU(u)\mu(u) + \frac{\text{tr} [\sigma(u)\sigma^*(u)D^2U(u)]}{2} + \frac{\|\sigma^*(u)DU(u)\|^2}{2e^{\alpha t}} \leq \alpha_1 U(u) + \beta_1$$

with  $\alpha_1 = 2C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2$  and  $\beta_1 = \frac{C_\delta^2}{2} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 (\|u_0\|_H^2 + \|u_0\|_H^8)$ . Applying Lemma 3.1 with  $\bar{U} = -\beta_\lambda$  for  $\lambda = \pm 1$ , we obtain

$$(3.8) \quad \sup_{t \in [0, T]} \mathbb{E} \left[ \exp \left( \frac{U(u(t))}{e^{\alpha_\lambda t}} - \int_0^t \frac{\beta_\lambda}{e^{\alpha_\lambda s}} ds \right) \right] \leq \mathbb{E} [e^{U(u_0)}].$$

When  $\lambda = -1$ , (3.8) yields that

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \exp \left( \frac{U(u(t))}{e^{\alpha_{-1} t}} \right) \right] \leq e^{\frac{1}{2}} \mathbb{E} [e^{U(u_0)}].$$

Applying Young inequality and a version of Jensen inequality, we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0\|_H \|\nabla u\|_H ds \right) \right\|_{L^p(\Omega)} \\ & \leq \left\| \exp \left( \frac{T}{2\epsilon} \|u_0\|_H^2 \right) \right\|_{L^{2p}(\Omega)} \left\| \exp \left( \int_0^T \frac{\epsilon}{2} \|\nabla u\|_H^2 ds \right) \right\|_{L^{2p}(\Omega)} \\ & \leq \left\| \exp \left( \frac{T}{2\epsilon} \|u_0\|_H^2 \right) \right\|_{L^{2p}(\Omega)} \sup_{t \in [0, T]} \left\| e^{T\epsilon U(u(t))} \right\|_{L^{2p}(\Omega)}. \end{aligned}$$

Let  $\epsilon = \frac{1}{2pTe^{\alpha_{-1}T}}$ , then

$$\begin{aligned} & \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0\|_H \|\nabla u\|_H ds \right) \right\|_{L^p(\Omega)} \\ & \leq e^{\frac{1}{4p}} \sqrt[2p]{\mathbb{E} \left[ \exp \left( 2p^2 T^2 e^{C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 T} \|u_0\|_H^2 \right) \right]} \sqrt[2p]{\mathbb{E} [e^{U(u_0)}]} \\ & \leq e^{\frac{1}{4p}} \sqrt[4p]{\mathbb{E} [e^{\|u_0\|_H^4}]} \sqrt[4p]{\mathbb{E} \left[ \exp \left( 4p^4 T^4 e^{2C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 T} \right) \right]} \sqrt[2p]{\mathbb{E} [e^{U(u_0)}]}. \end{aligned}$$

When  $\lambda = 1$ , by the fact that  $U(u) \geq \frac{1}{4}(\|\nabla u\|_H^2 - \|u_0\|_H^6)$  and a version of Jensen inequality, we have

$$\begin{aligned}
& \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0\|_H \|\nabla u\|_H ds \right) \right\|_{L^p(\Omega)} \\
& \leq \left\| \exp \left( \int_0^T \frac{\|u_0\|_H^2}{\epsilon} + \frac{\epsilon \|u_0\|_H^6}{4} + \frac{\beta_1}{2pe^{\alpha_1 r}} dr \right) \right\|_{L^{2p}(\Omega)} \\
& \quad \times \left\| \exp \left( \int_0^T \frac{\epsilon(\|\nabla u\|_H^2 - \|u_0\|_H^6)}{4} - \frac{\beta_1}{2pe^{\alpha_1 r}} dr \right) \right\|_{L^{2p}(\Omega)} \\
& \leq \left\| \exp \left( \frac{T\|u_0\|_H^2}{\epsilon} + \frac{\epsilon T\|u_0\|_H^6}{4} + \frac{\beta_1(1 - e^{-\alpha_1 T})}{2p\alpha_1} \right) \right\|_{L^{2p}(\Omega)} \\
& \quad \times \left\| \exp \left( \epsilon T U(u(t)) - \frac{\beta_1(1 - e^{-\alpha_1 T})}{2p\alpha_1} \right) \right\|_{L^{2p}(\Omega)}.
\end{aligned}$$

Let  $\epsilon = \frac{1}{2pTe^{\alpha_1 T}}$ , then by Young inequality,

$$\begin{aligned}
& \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0\|_H \|\nabla u\|_H ds \right) \right\|_{L^p(\Omega)} \\
& \leq \sqrt[2p]{\mathbb{E} \left[ \exp \left( 4p^2 T^2 e^{\alpha_1 T} \|u_0\|_H^2 + \frac{\|u_0\|_H^6}{4e^{\alpha_1 T}} + \frac{\beta_1(1 - e^{-\alpha_1 T})}{\alpha_1} \right) \right]} \sqrt[2p]{\mathbb{E} [e^{U(u_0)}]} \\
& \leq e^{\frac{1}{8p}} \sqrt[4p]{\mathbb{E} [e^{\|u_0\|_H^6 + 4\|u_0\|_H^4}]} \sqrt[4p]{\mathbb{E} \left[ \exp \left( 4p^4 T^4 e^{4C_\delta^2 \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2 \|u_0\|_H^2 T} \right) \right]} \sqrt[2p]{\mathbb{E} [e^{U(u_0)}]}.
\end{aligned}$$

This shows (3.6).

For the discrete case, simple calculations show that

$$\begin{aligned}
(DU^h(X^h))Y^h &= \langle D_+ X^h, D_+ Y^h \rangle_h - \lambda \langle |X^h|^2 X^h, Y^h \rangle_h, \\
(D^2 U^h(X^h))(Y^h, Z^h) &= \langle D_+ Z^h, D_+ Y^h \rangle_h - \lambda \langle |X^h|^2 Y^h, Z^h \rangle_h \\
&\quad - 2\lambda \langle \Re [\bar{X}^h Y^h] X^h, Z^h \rangle_h.
\end{aligned}$$

In the case of Eq. (1.5),  $\mu^h(u^h)(l) = \mathbf{i}D_+ D_- u^h(l) + \mathbf{i}\lambda |u^h(l)|^2 u^h(l) - u^h(l)F_Q(l)/2$  and  $\sigma^h(u^h)(e_k)(l) = -\mathbf{i}u^h(l)Q^{\frac{1}{2}}e_k(l)$ . Then

$$\begin{aligned}
DU^h(u^h)\mu^h(u^h) &= -\frac{1}{2} \sum_{l=0}^N |D_+ u^h(l)|^2 F_Q(l+1)h - \frac{1}{2} \langle D_+ u^h, u^h D_+ F_Q \rangle_h + \frac{\lambda}{2} \langle |u^h|^4, F_Q \rangle_h, \\
\text{tr} \left[ \sigma^h(u^h) \sigma^h(u^h)^* (D^2 U^h)(u^h) \right] &= \sum_{l=0}^N |D_+ u^h(l)|^2 F_Q(l+1)h - \lambda \langle |u^h|^4, F_Q \rangle_h \\
&\quad + 2 \sum_{k=1}^{\infty} \langle D_+ u^h, u^h (Q^{\frac{1}{2}} e_k) D_+ (Q^{\frac{1}{2}} e_k) \rangle_h + \sum_{k=1}^{\infty} \langle |u^h|^2, |D_+ (Q^{\frac{1}{2}} e_k)|^2 \rangle_h
\end{aligned}$$

and

$$\|\sigma^h(u^h)^* DU^h(u^h)\|_h^2 = \sum_{k=1}^{\infty} \langle D_+ u^h, -\mathbf{i}u^h D_+ (Q^{\frac{1}{2}} e_k) \rangle_h^2.$$

As a consequence,

$$\begin{aligned}
& DU^h(u^h)\mu^h(u^h) + \frac{1}{2}\text{tr} [\sigma^h(u^h)\sigma^h(u^h)^*(D^2U^h)(u^h)] + \frac{1}{2e^{\alpha t}}\|\sigma^h(u^h)^*DU^h(u^h)\|_h^2 \\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\langle D_+u^h, u^h|D_+(Q^{\frac{1}{2}}e_k)|^2h\rangle_h + \frac{1}{2}\sum_{k=1}^{\infty}\langle |u^h|^2, |D_+(Q^{\frac{1}{2}}e_k)|^2\rangle_h \\
&\quad + \frac{1}{2e^{\alpha t}}\sum_{k=1}^{\infty}\langle D_+u^h, -iu^hD_+(Q^{\frac{1}{2}}e_k)\rangle_h^2 \\
&\leq \frac{3C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2 + \frac{C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2\|D_+u^h\|_h^2.
\end{aligned}$$

We conclude that, for  $\lambda = -1$ ,

$$\begin{aligned}
& DU^h(u^h)\mu^h(u^h) + \frac{\text{tr} [\sigma^h(u^h)\sigma^h(u^h)^*(D^2U^h)(u^h)]}{2} \\
&\quad + \frac{\|\sigma^h(u^h)^*DU^h(u^h)\|^2}{2e^{\alpha h t}} \leq \alpha_{-1}^h U^h(u^h) + \beta_{-1}^h
\end{aligned}$$

with  $\alpha_{-1}^h = C_\delta^2\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2$  and  $\beta_{-1}^h = \frac{3C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2$ . Similarly to the continuous case,

$$\begin{aligned}
& \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0^h\|_h \|D_+u^h\|_h ds \right) \right\|_{L^p(\Omega)} \\
&\leq \left\| \exp \left( e^{\alpha_{-1}^h T} p T^2 \|u_0^h\|_h^2 \right) \right\|_{L^{2p}(\Omega)} e^{\frac{1}{4p}} \sqrt[p]{\mathbb{E} [e^{U^h(u_0^h)}]} \\
&\leq e^{\frac{1}{4p}} \sqrt[p]{\mathbb{E} [e^{\|u_0^h\|_h^4}]} \sqrt[p]{\mathbb{E} \left[ \exp \left( 4p^4 T^4 e^{\frac{2C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2 T} \right) \right]} \sqrt[p]{\mathbb{E} [e^{U^h(u_0^h)}]}.
\end{aligned}$$

When  $\lambda = 1$ , Similar arguments yield that

$$\begin{aligned}
& DU^h(u^h)\mu^h(u^h) + \frac{\text{tr} [\sigma^h(u^h)\sigma^h(u^h)^*(D^2U^h)(u^h)]}{2} \\
&\quad + \frac{1}{2e^{\alpha_1^h t}}\|\sigma^h(u^h)^*DU^h(u^h)\|^2 \leq \alpha_1^h U^h(u^h) + \beta_1^h
\end{aligned}$$

with  $\alpha_1^h = 2C_\delta^2\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0^h\|_h^2$  and  $\beta_1^h = \frac{C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2(3\|u_0\|_H^2 + \|u_0\|_H^8)$ . Moreover, we have

$$\begin{aligned}
& \sup_{t \in [0, T]} \left\| \exp \left( \int_0^t \|u_0^h\|_h \|D_+u^h\|_h ds \right) \right\|_{L^p(\Omega)} \\
&\leq \sqrt[p]{\mathbb{E} \left[ \exp \left( 4p^2 T^2 e^{\alpha_1^h T} \|u_0^h\|_h^2 + \frac{\|u_0^h\|_h^6}{4e^{\alpha_1^h T}} + \frac{\beta_1^h(1 - e^{-\alpha_1^h T})}{\alpha_1^h} \right) \right]} \sqrt[p]{\mathbb{E} [e^{U^h(u_0^h)}]} \\
&\leq e^{\frac{3}{8p}} \sqrt[p]{\mathbb{E} [e^{\|u_0^h\|_h^6 + 4\|u_0^h\|_h^4}]} \sqrt[p]{\mathbb{E} \left[ \exp \left( 4p^4 T^4 e^{\frac{4C_\delta^2}{2}\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}^2\|u_0\|_H^2 T} \right) \right]} \sqrt[p]{\mathbb{E} [e^{U^h(u_0^h)}]}.
\end{aligned}$$

This shows (3.7) and we complete the proof.  $\square$

The above proposition implies that the solution of the stochastic cubic Schrödinger equation (1.1) is continuously depending on the initial data.

COROLLARY 3.1. Let  $Q^{\frac{1}{2}} \in \mathcal{L}_2^2$  and (3.4) hold for  $u_0$  and  $v_0$  with  $p = 8$ . Let  $u = \{u(t) : t \in [0, T]\}$  and  $v = \{v(t) : t \in [0, T]\}$  be the solutions of Eq. (1.1) with initial data  $u_0$  and  $v_0$ , respectively. Then there exists  $C \in (0, \infty)$  depending on  $T$ ,  $\mathbb{E}[e^{U(u_0)}]$ ,  $\mathbb{E}[e^{U(v_0)}]$  and  $\|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}$  such that

$$(3.9) \quad \sqrt{\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t) - v(t)\|_H^2 \right]} \leq C \sqrt[4]{\mathbb{E} [\|u_0 - v_0\|_H^4]}.$$

PROOF. Applying Itô isometry to the functional  $f(X(t)) := \|X(t)\|_H^2$  with  $X(t) = u(t) - v(t)$ , we obtain

$$\begin{aligned} & \|u(t) - v(t)\|_H^2 \\ &= \|u_0 - v_0\|_H^2 + \int_0^t \sum_{k=1}^{\infty} \|\mathbf{i}(u-v)Q^{\frac{1}{2}}e_k\|_H^2 dr + 2 \int_0^t \langle u-v, -\mathbf{i}(u-v) \rangle_H dW(s) \\ & \quad + 2 \int_0^t \langle u-v, \mathbf{i}\Delta(u-v) + \mathbf{i}\lambda(|u|^2u - |v|^2v) - (u-v)F_Q/2 \rangle_H dr \\ & \leq \|u_0 - v_0\|_H^2 + \int_0^t 2\|u\|_{L^\infty}\|v\|_{L^\infty}\|u-v\|_H^2 dr. \end{aligned}$$

Applying Gronwall inequality and taking expectation, we get

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t) - v(t)\|_H^2 \right] \leq \left\| \exp \left( \int_0^T 2\|u\|_{L^\infty}\|v\|_{L^\infty} dr \right) \right\|_{L^2(\Omega)} \sqrt{\mathbb{E} [\|u_0 - v_0\|_H^4]}.$$

We conclude (3.9) by Cauchy-Schwarz inequality, (1.2) and Proposition 3.1.  $\square$

We also have the following large derivation-type result on

$$(3.10) \quad \mathbf{i}du + \Delta udt + \lambda|u|^2udt = \epsilon u \circ dW, \quad \epsilon \in \mathbb{R}$$

with explicit strong convergence rate.

COROLLARY 3.2. Let  $Q^{\frac{1}{2}} \in \mathcal{L}_2^2$  and (3.4) hold with  $p = 8$ . Let  $u^\epsilon = \{u^\epsilon(t) : t \in [0, T]\}$  and  $u^0 = \{u^0(t) : t \in [0, T]\}$  be the solutions of Eq. (3.10) with the same initial datum  $u_0$ , respectively. Then there exists  $C = C(T, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^{1+\delta}}) \in (0, \infty)$  such that

$$(3.11) \quad \sqrt{\mathbb{E} \left[ \sup_{t \in [0, T]} \|u^\epsilon(t) - u^0(t)\|_H^2 \right]} \leq C|\epsilon|.$$



PROOF. Applying Itô isometry to the functional  $f(X(t)) := \|X(t)\|_H^2$  with  $X(t) = u^\epsilon(t) - u^0(t)$ , we obtain

$$\begin{aligned} & \|u^\epsilon(t) - u^0(t)\|_H^2 \\ &= \epsilon^2 \int_0^t \sum_k \|u^\epsilon Q^{\frac{1}{2}} e_k\|_H^2 dr + 2 \int_0^t \langle u^\epsilon - u^0, -\mathbf{i}(u^\epsilon - u^0) \rangle_H dW(s) \\ & \quad + 2 \int_0^t \langle u^\epsilon - u^0, \mathbf{i}\Delta(u^\epsilon - u^0) + \mathbf{i}\lambda(|u^\epsilon|^2 u^\epsilon - |u^0|^2 u^0) - \epsilon^2 u^\epsilon F_Q/2 \rangle_H dr \\ & \leq \epsilon^2 \int_0^t \sum_k \langle u^0, u^\epsilon |Q^{\frac{1}{2}} e_k|^2 \rangle_H dr + \int_0^t 2\|u^\epsilon\|_{L^\infty} \|u^0\|_{L^\infty} \|u^\epsilon - u^0\|_H^2 dr. \end{aligned}$$

Applying Gronwall-Bellman inequality, taking expectation and using Sobolev embedding, we get

$$\begin{aligned} & \sqrt{\mathbb{E} \left[ \sup_{t \in [0, T]} \|u^\epsilon(t) - u^0(t)\|_H^2 \right]} \\ & \leq \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^1} \|u_0\|_{L^4(\Omega; H)} \left\| \exp \left( \int_0^T \|u^\epsilon\|_{L^\infty} \|u^0\|_{L^\infty} dr \right) \right\|_{L^4(\Omega)} T^{\frac{1}{2}} |\epsilon|. \end{aligned}$$

Applying (1.2) and Hölder inequality, we conclude (3.11) by Proposition 3.1.  $\square$

REMARK 3.1. For stochastic NLS equation driven by an additive noise, Corollary 3.1 and Proposition 3.5 in [9] derived a.s. continuous dependence on the initial data and the noise without rate.

#### 4. Strong convergence rate of finite difference approximation

In this section, we establish the strong convergence rate of the spatial center difference scheme

$$(4.1) \quad du^h(l) = \left( \mathbf{i}D_+ D_- u^h(l) + \mathbf{i}\lambda |u^h(l)|^2 u^h(l) - u^h(l) F_Q(l)/2 \right) dt - \mathbf{i}u^h(l) dW(t, l)$$

for Eq. (1.1). It is clear that the exact solution of Eq. (1.1), at the grid points, satisfies

$$(4.2) \quad du(l) = \left( \mathbf{i}D_+ D_- u(l) + \mathbf{i}R_h(l) + \mathbf{i}\lambda |u(l)|^2 u(l) - u(l) F_Q(l)/2 \right) dt - \mathbf{i}u(l) dW(t, l),$$

where  $R_h(l) := \Delta u(l) - D_+ D_- u(l)$  for  $l \in \{0, 1, \dots, N, N+1\}$ .

Denote by  $\epsilon^h(t, l)$  the difference between  $u^h(t)$  determined by (4.1) and  $u(t)$  determined by Eq. (4.2) at grid point  $x_l$  for  $t \in [0, T]$  and  $l \in \{0, 1, \dots, N, N+1\}$ . Our main result is the following error estimate of  $\epsilon^h$  in strong sense.

THEOREM 4.1. Assume that  $Q^{\frac{1}{2}} \in \mathcal{L}_2^5$ ,  $u_0 \in \bigcap_{m=2}^5 L^{4 \cdot 3^{5-m}}(\Omega; H^m)$  and (3.4)–(3.5) hold for  $p = 8$ . There exists  $C = C(T, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^5})$  such that

$$(4.3) \quad \sqrt{\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t) - u^h(t)\|_h^2 \right]} \leq Ch^2.$$

PROOF. Subtracting Eq. (4.1) from Eq. (4.2) and using the identity  $|a|^2 a - |b|^2 b = (|a|^2 + |b|^2)(a - b) + ab \overline{a - b}$  for  $a, b \in \mathbb{C}$ , we obtain

$$\begin{aligned} d\epsilon^h(l) = & \left( \mathbf{i}D_+ D_- \epsilon^h(l) + \mathbf{i}\lambda(|u(l)|^2 + |u^h(l)|^2)\epsilon^h(l) + \mathbf{i}\lambda u(l)u^h(l)\overline{\epsilon^h(l)} \right) dt \\ & - \epsilon^h(l)F_Q(l)/2dt + \mathbf{i}R_h(l)dt - \mathbf{i}\epsilon^h(l)dW(t, l). \end{aligned}$$

Applying Itô formula to the functional  $f(X(t)) := \|X(t)\|_h^2$  with  $X(t) = \epsilon_N^h(t)$  and using the fact that  $\epsilon_N^h(0, l) = 0$  for any  $l \in \{0, 1, \dots, N\}$  and the symmetry of the proposed scheme on Dirichlet boundary condition, we get

$$\sum_{l=0}^N |\epsilon^h(t, l)|^2 = 2\lambda \int_0^t \sum_{l=0}^N \Re \left[ \mathbf{i}u(l)u^h(l)\overline{\epsilon^h(l)} \right] dr + 2 \int_0^t \sum_{l=0}^N \Re \left[ \overline{\mathbf{i}\epsilon^h(l)} R^h(l) \right] dr,$$

Applying Cauchy-Schwarz inequality and the discrete Gagliardo-Nirenberg inequality (1.3), we obtain

$$\begin{aligned} \|\epsilon^h(t)\|_h^2 & \leq \int_0^t \|R^h\|_\infty^2 dr + \int_0^t (1 + \|u\|_{L^\infty}^2 + \|u^h\|_\infty^2) \|\epsilon^h\|_h^2 dr \\ & \leq \int_0^t \|R^h\|_\infty^2 dr + \int_0^t (1 + 2\|u_0\|_H \|\nabla u\|_H + 2\|u_0^h\|_h \|D_+ u^h\|_h) \|\epsilon^h\|_h^2 dr. \end{aligned}$$

It follows from Gronwall-Bellman inequality that

$$\|\epsilon^h(t)\|_h^2 \leq \left( \int_0^T \|R^h\|_\infty^2 dr \right) \exp \left( \int_0^T 1 + 2\|u_0\|_H \|\nabla u\|_H + 2\|u_0^h\|_h \|D_+ u^h\|_h dr \right).$$

Taking expectation and using Hölder and Minkovskii inequalities, we derive

$$(4.4) \quad \left\| \exp \left( \int_0^T \|u_0\|_H \|\nabla u\|_H dr \right) \right\|_{L^8(\Omega)} \left\| \exp \left( \int_0^T \|u_0^h\|_h \|D_+ u^h\|_h dr \right) \right\|_{L^8(\Omega)} \sqrt{\mathbb{E} \left[ \sup_{t \in [0, T]} \|\epsilon^h(t)\|_h^2 \right]} \leq T^{\frac{1}{2}} e^{\frac{T}{2}} \sqrt[4]{\mathbb{E} \left[ \sup_{t \in [0, T]} \|R^h\|_\infty^4 \right]}.$$

We proceed with bounding the three expectations on the right hand side of the above inequality.

By Taylor expansion, there exists  $\theta_l \in [(l-1)h, (l+1)h]$  such that  $R_h(l) := u_{xxxx}(\theta_l)h^2/4!$ . Combining Sobolev embedding, Corollary 2.1 implies, with  $p = 4$  and  $s = 5$ , that

$$(4.5) \quad \sqrt[4]{\mathbb{E} \left[ \sup_{t \in [0, T]} \|R^h\|_\infty^4 \right]} \leq C \sqrt[4]{\mathbb{E} \left[ \sup_{t \in [0, T]} |u|_{H^5}^4 \right]} h^2 \leq Ch^2.$$

By (3.6) and (3.7) in Proposition 3.1, we get

$$\begin{aligned} & \left\| \exp \left( \int_0^T \|u_0\|_H \|\nabla u\|_H dr \right) \right\|_{L^8(\Omega)} \left\| \exp \left( \int_0^T \|u_0^h\|_h \|D_+ u^h\|_h dr \right) \right\|_{L^8(\Omega)} \\ (4.6) \quad & \leq C \sqrt[16]{\mathbb{E} [e^{U(u_0)}]} \sqrt[16]{\mathbb{E} [e^{U^h(u_0^h)}]}. \end{aligned}$$

We conclude (4.3) by combining (4.4)–(4.6).  $\square$

COROLLARY 4.1. Assume that  $Q^{\frac{1}{2}} \in \mathcal{L}_2^5$ ,  $u_0 \in \bigcap_{m=2}^5 L^{2p \cdot 3^{5-m}}(\Omega; H^m)$  for some  $p \in [2, \infty)$  and (3.4)–(3.5) hold for  $4p$ . Then there exists  $C = C(T, p, u_0, \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2^5})$  such that

$$(4.7) \quad \sqrt[p]{\mathbb{E} \left[ \sup_{t \in [0, T]} \|u(t) - u^h(t)\|_h^p \right]} \leq Ch^2.$$

PROOF. Similarly to the proof in Theorem 4.1, we apply Itô formula to  $\|\epsilon(t)\|_h^p$ , combined with Young inequality, and obtain

$$\begin{aligned} \|\epsilon(t)\|_h^p &\leq \|\epsilon(0)\|_h^p + \int_0^t p \|\epsilon(r)\|_h^{p-1} (\|u_0\|_H \|\nabla u\|_H + \|u_0^h\|_h \|D_+ u^h\|_h) dr \\ &\quad + \int_0^t p \|\epsilon(r)\|_h^{p-1} \|R^h\|_\infty dr \\ &\leq \|\epsilon(0)\|_h^p + \int_0^t \|R^h\|_\infty^p dr + \int_0^t \|\epsilon(r)\|_h^p (p-1 + p\|u_0\|_H \|\nabla u\|_H \\ &\quad + p\|u_0^h\|_h \|D_+ u^h\|_h) dr. \end{aligned}$$

The fact that  $\|\epsilon(0)\|_h = 0$  and Gronwall inequality yield that

$$\begin{aligned} \|\epsilon(t)\|_h^p &\leq \exp((p-1)T) \left( \int_0^T \|R^h\|_\infty^p dr \right) \\ &\quad \times \exp \left( \int_0^t p \|u_0\|_H \|\nabla u\|_H + p \|u_0^h\|_h \|D_+ u^h\|_h dr \right). \end{aligned}$$

We conclude (4.7) by similar proof of (4.3) in Theorem 4.1.  $\square$

- REMARK 4.1. (1) Our error analysis is also available under rough regularity assumptions. More precisely, for some  $\delta > 1/2$  and  $0 < \beta < 1$ , if  $u \in H^{2+\beta+\delta}$  a.s., then  $\|R_h\|_\infty \leq C\|u\|_{2+\beta+\delta} h^\beta$ , where  $\|\cdot\|_{C^\beta}$  is the usual Hölder norm, which implies that the strong convergence error is order  $\mathcal{O}(h^\beta)$ ; if  $u \in H^{3+\beta+\delta}$  a.s., then  $\|R_h\|_\infty \leq C\|u\|_{3+\beta+\delta} h^{1+\beta}$  which yields that the strong convergence error is order  $\mathcal{O}(h^{1+\beta})$ .
- (2) If one have a priori estimate of  $u$  under the  $H^\delta$  norm for some  $\delta > 0$ , we can also obtain strong convergence rate for our scheme (1.5) under weak regularity assumptions. For example, once a priori estimate under the  $H^{4+\delta}$ -norm with  $\delta > 1/2$  is established, we can reduce the regularity condition  $H^5$  in Theorem 4.1 to  $H^{4+\delta}$ . When the regularity exceedr  $H^{4+\delta}$  with  $\delta > 1/2$ , the order of the scheme (1.5) can not be improved. In this case, to obtain higher order schemes one can consider other schemes or use the extrapolation acceleration skill (see e.g. [14]).

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